https://www.linkedin.com/feed/update/urn:li:activity:6506784687548174336

3970. Proposed by Nermin Hodžić and Salem Malikić.

Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{10a^3+9}+\frac{1}{10b^3+9}+\frac{1}{10c^3+9}\ \geq\ \frac{3}{19}.$$

Solution by Arkady Alt , San Jose ,California, USA.

★ First we will prove that for any $x,y \geq 0$ such that $x+y \geq \dfrac{3}{\sqrt[3]{5}}$ holds inequality

(**1**) $$\frac{1}{10x^3+9}+\frac{1}{10y^3+9} \geq \frac{2}{10\left(\frac{x+y}{2}\right)^3+9}.$$

Let $p := x+y, q := xy$. Then

$0 \leq q \leq p^2/4,\ (10x^3+9)(10y^3+9) = 100x^3y^3+90(x^3+y^3)+81 =$
$100q^3+90p(p^2-3q)+81 = 81-270pq+90p^3+100q^3 \geq 81$ and, therefore,

$$\frac{1}{10x^3+9}+\frac{1}{10y^3+9}-\frac{2}{10\left(\frac{x+y}{2}\right)^3+9} = \frac{2(5p(p^2-3q)+9)}{100q^3+90p(p^2-3q)+81}-\frac{8}{5p^3+36} =$$

$$\frac{10(p^2-4q)(20q^2+5p^2q+5p^4-27p)}{(5p^3+36)(81-270pq+90p^3+100q^3)} \geq 0$$

because $20q^2+5p^2q+5p^4-27p \geq p(5p^3-27) \geq 0$.

Assuming WLOG that $a+b \geq 2 > \dfrac{3}{\sqrt[3]{5}}$ and by replacing (x,y) in (**1**) with (a,b)

we obtain $$\frac{1}{10a^3+9}+\frac{1}{10b^3+9} \geq \frac{2}{10\left(\frac{a+b}{2}\right)^3+9} = \frac{8}{5(3-c)^3+36}.$$

Therefore, $$\sum\frac{1}{10a^3+9}-\frac{3}{19} \geq \frac{8}{5(3-c)^3+36}+\frac{1}{10c^3+9}-\frac{3}{19} =$$

$$\frac{30c(60c+5c^3+36-35c^2)(c-1)^2}{19(10c^3+9)\left(5(3-c)^3+36\right)} \geq 0$$ because $0 \leq c = 3-a-b \leq 1$

implies $c(60c+5c^3+36-35c^2) \geq 0$.